

# On finite determinacy for matrices of power series

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## Abstract

Let  $R = K[[x_1, \dots, x_s]]$  be the ring of formal power series with maximal ideal  $\mathfrak{m}$  over a field  $K$  of arbitrary characteristic. On the ring  $M_{m,n}$  of  $m \times n$  matrices  $A$  with entries in  $R$  we consider several equivalence relations given by the action on  $M_{m,n}$  of a group  $G$ .  $G$  can be the group of automorphisms of  $R$ , combined with the multiplication of invertible matrices from the left, from the right, or from both sides, respectively. We call  $A$  finitely  $G$ -determined if  $A$  is  $G$ -equivalent to any matrix  $B$  with  $A - B \in \mathfrak{m}^k M_{m,n}$  for some finite integer  $k$ , which implies in particular that  $A$  is  $G$ -equivalent to a matrix with polynomial entries.

The classical criterion for analytic or differential map germs  $f : (K^s, 0) \rightarrow (K^m, 0)$ ,  $K = \mathbb{R}, \mathbb{C}$ , says that  $f \in M_{m,1}$  is finitely determined (with respect to various group actions) iff the tangent space to the orbit of  $f$  has finite codimension in  $M_{m,1}$ . We extend this criterion to arbitrary matrices in  $M_{m,n}$  if the characteristic of  $K$  is 0 or, more general, if the orbit map is separable. In positive characteristic however, the problem is more subtle since the orbit map is in general not separable, as we show by an example. This fact had been overlooked in previous papers. Our main result is a general sufficient criterion for finite  $G$ -determinacy in  $M_{m,n}$  in arbitrary characteristic in terms of the tangent image of the orbit map, which we introduce in this paper. This criterion provides a computable bound for the  $G$ -determinacy of a matrix  $A$  in  $M_{m,n}$ , which is new even in characteristic 0.

## 1 Overview

Throughout this paper let  $K$  denote a field of arbitrary characteristic and

$$R := K[[\mathbf{x}]] = K[[x_1, x_2, \dots, x_s]]$$

the formal power series ring over  $K$  with maximal ideal  $\mathfrak{m}$ . We denote by

$$M_{m,n} := \text{Mat}(m, n, R)$$

the ring of all  $m \times n$  matrices of power series. We consider the group of  $K$ -algebra automorphisms of  $R$

$$\mathcal{R} := \text{Aut}(R)$$

and the semi-direct products

$$\begin{aligned}\mathcal{G}_l &:= GL(m, R) \rtimes \mathcal{R}, \\ \mathcal{G}_r &:= GL(n, R) \rtimes \mathcal{R}, \text{ and} \\ \mathcal{G}_{lr} &:= (GL(m, R) \times GL(n, R)) \rtimes \mathcal{R}.\end{aligned}$$

These groups act on the space  $M_{m,n}$  as follows

$$\begin{aligned}(\phi, A) &\mapsto \phi(A) := [\phi(a_{ij}(\mathbf{x}))] = [a_{ij}(\phi(\mathbf{x}))], \\ (U, \phi, A) &\mapsto U \cdot \phi(A) = U \cdot [\phi(a_{ij}(\mathbf{x}))] = U \cdot [a_{ij}(\phi(\mathbf{x}))], \\ (V, \phi, A) &\mapsto \phi(A) \cdot V = [\phi(a_{ij}(\mathbf{x}))] \cdot V = [a_{ij}(\phi(\mathbf{x}))] \cdot V, \text{ and} \\ (U, V, \phi, A) &\mapsto U \cdot \phi(A) \cdot V = U \cdot [\phi(a_{ij}(\mathbf{x}))] \cdot V = U \cdot [a_{ij}(\phi(\mathbf{x}))] \cdot V,\end{aligned}$$

where  $\mathbf{x} = (x_1, x_2, \dots, x_s)$ ,  $A = [a_{ij}(\mathbf{x})] \in M_{m,n}$ ,  $U \in GL(m, R)$ ,  $V \in GL(n, R)$ , and  $\phi(\mathbf{x}) := (\phi_1, \dots, \phi_s)$  with  $\phi_i := \phi(x_i) \in \mathfrak{m}$  for all  $i = 1, \dots, s$ .

Throughout this paper let  $G$  denote one of the groups  $\mathcal{R}$ ,  $\mathcal{G}_l$ ,  $\mathcal{G}_r$ , and  $\mathcal{G}_{lr}$ .

For  $A \in M_{m,n}$ , we denote by  $GA$  the orbit of  $A$  under the action of  $G$  on  $M_{m,n}$ . Two matrices  $A, B \in M_{m,n}$  are called  $G$ -equivalent, denoted  $A \stackrel{G}{\sim} B$ , if  $B \in GA$ . A matrix  $A \in M_{m,n}$  is said to be  $G$   $k$ -determined if for each matrix  $B \in M_{m,n}$  with  $B - A \in \mathfrak{m}^{k+1} \cdot M_{m,n}$ , we have  $B \stackrel{G}{\sim} A$ , i.e. if  $A$  is  $G$ -equivalent to every matrix which coincides with  $A$  up to and including terms of degree  $k$ .  $A$  is called *finitely  $G$ -determined* if there exists a positive integer  $k$  such that it is  $G$   $k$ -determined.

Note that the case  $n = 1$ , i.e.  $M_{m,1}$ , covers the case of map-germs  $(f_1, \dots, f_m)$ ,  $K[[y_1, \dots, y_m]] \rightarrow R$ ,  $y_i \mapsto f_i$ , where  $G$ -equivalence is called *right-equivalence* for  $G = \mathcal{R}$  and *contact-equivalence* for  $G = \mathcal{G}_l$ ; the case  $m = n = 1$  is the classical case of one power series. In [GP16] we give necessary and sufficient conditions for finite determinacy of map germs in arbitrary characteristic, in particular for complete intersections, also for non-separable orbit maps.

Over the real and complex numbers  $K$ , finite determinacy was studied for  $M_{m,1}$ , i.e. for differentiable and analytic map-germs  $(f_1, \dots, f_m) : (K^m, 0) \rightarrow (K^s, 0)$ , by [Tou68], [Mat68], [Wal81], [Gaf79], [dP80], [Dam81], [BdPW87],  $\dots$ . In [BK16], the authors study finite determinacy for matrices of power series in  $M_{m,n}$  over fields of characteristic 0 with respect to various equivalence relations. As the methods of proof usually involve integration of vector fields, they can not be transferred to positive characteristic. Moreover, the

starting point of all previous investigations was the fact that the tangent space at a point of the orbit is equal to the image of the tangent space of the group at the identity under the orbit map. This is in general no more true in positive characteristic as we show in this paper. The case of one power series, i.e.  $M_{1,1}$ , over a field of arbitrary characteristic was treated in [GK90] for contact equivalence and in [BGM12] for right and contact equivalence. The present paper is an extension of some results of the PhD thesis of the second author, see [P16].

We give a short overview of the results of this paper: In section 2 we introduce the *tangent image*  $\tilde{T}_A(GA)$  at a matrix  $A$  to the orbit  $GA$  to be the inverse limit of the images of the tangent maps to

$$o^{(k)} : G^{(k)} \rightarrow G^{(k)} \text{jet}_k(A),$$

where  $o^{(k)}$  is the induced map by restricting the orbit map  $G \rightarrow GA$  to the jet space of power series up to order  $k$ . In Proposition 2.5 and Definition 2.6 we give an explicit and computable description of  $\tilde{T}_A(GA)$  in terms of the entries and the partials of the entries of  $A$ .

In section 3, Theorem 3.2, we prove our main result:

**Theorem:** *Let  $A \in \mathfrak{m} \cdot M_{m,n}$ . If*

$$\dim_K \left( M_{m,n} / \tilde{T}_A(GA) \right) < \infty,$$

*then  $A$  is finitely  $G$ -determined. More precisely, if there is some integer  $k \geq 0$  such that*

$$\mathfrak{m}^{k+2} \cdot M_{m,n} \subset \mathfrak{m} \cdot \tilde{T}_A(GA),$$

*then  $A$  is  $G(2k - \text{ord}(A) + 2)$ -determined.*

Here, for  $f \in R$ ,  $f \neq 0$ , we denote by  $\text{ord}(f)$  the order of  $f$ , i.e. the maximal positive integer  $l$  such that  $f \in \mathfrak{m}^l$ , and for  $f = 0$ , we set  $\text{ord}(f) = \infty$ . For  $A = [a_{ij}] \in M_{m,n}$ , we set  $\text{ord}(A) := \min\{\text{ord}(a_{ij})\}$ .

In section 4, we formulate several other equivalent sufficient conditions for finite  $G$ -determinacy and prove alternative determinacy bounds. If the orbit map  $o^{(k)}$  is separable for sufficiently big  $k$ , these conditions are even equivalent to finite determinacy as we prove in Theorem 4.3:

**Theorem:** *Assume there is some  $k \in \mathbb{N}$  such that the orbit map  $G^{(l)} \rightarrow G^{(l)} \text{jet}_l(A)$  is separable for all  $l \geq k$  (e.g. if  $\text{char}(K) = 0$ ). Then  $A$  is finitely  $G$ -determined if and only if  $\tilde{T}_A(GA)$  has finite codimension in  $M_{m,n}$ .*

This follows since for a separable orbit map the tangent map is surjective and hence the tangent image to the orbit coincides with the tangent space to the orbit. We show

by an example (cf. Example 2.9) that even in the simplest case of one function  $f$  the orbit map need not be separable if  $K$  has positive characteristic, also for  $f$  being finitely determined. This fact had been overlooked in previous papers and came as a surprise to us, see Remark 2.10. Finally we apply the above result to classify finitely  $\mathcal{R}$ -determined matrices  $A \in M_{m,n}$  (under the separability condition) and show that for  $m > 1$  finite  $\mathcal{R}$ -determinacy holds only in the non-singular case.

## 2 The tangent image of the orbit map

In this section, we identify the images of the tangent maps induced by the orbit maps. Since the power series ring  $R$  and the group  $G \in \{\mathcal{R}, \mathcal{G}_l, \mathcal{G}_r, \mathcal{G}_{lr}\}$  are infinite dimensional over  $K$  we pass, as usual, to the space of  $k$ -jets.

For  $A \in M_{m,n}$  and  $k \in \mathbb{N}$ , we denote by  $jet_k(A)$  the image of  $A$  in  $M_{m,n}/\mathfrak{m}^{k+1} \cdot M_{m,n}$ , the  $k$ -jet of  $A$ , and by

$$M_{m,n}^{(k)} := M_{m,n}/\mathfrak{m}^{k+1} \cdot M_{m,n},$$

the space of all  $k$ -jets. The  $k$ -jet of  $G$  is

$$G^{(k)} := \{jet_k(g) \mid g \in G\},$$

where  $jet_k(g) = (jet_k(U), jet_k(V), jet_k(\phi))$ ,  $jet_k(\phi)(x_i) = jet_k(\phi(x_i))$ , for  $g = (U, V, \phi) \in \mathcal{G}_{lr}$  and similar for  $G = \mathcal{R}, \mathcal{G}_l, \mathcal{G}_r$ . Then  $G^{(k)} \in \{\mathcal{R}^{(k)}, \mathcal{G}_l^{(k)}, \mathcal{G}_r^{(k)}, \mathcal{G}_{lr}^{(k)}\}$  is an affine algebraic group with group structure given by  $jet_k(g)jet_k(h) = jet_k(gh)$ , acting algebraically on the affine space  $M_{m,n}^{(k)}$  via

$$G^{(k)} \times M_{m,n}^{(k)} \rightarrow M_{m,n}^{(k)}, (jet_k(g), jet_k(A)) \mapsto jet_k(gA),$$

i.e. we let representatives act and then take the  $k$ -jets. Everything is defined over  $K$ .

**Remark 2.1.** For a geometric interpretation of the orbit map and since we are going to apply results about algebraic group actions which are formulated for algebraically closed fields, we fix an algebraically closed extension field  $K'$  of  $K$ , e.g. an algebraic closure of  $K$ .

(1) Let  $H$  be an algebraic group defined over  $K$  acting  $K$ -algebraically on the algebraic  $K$ -variety  $X$ . Then  $X$  resp.  $H$  defines an algebraic variety  $X'$  resp. an algebraic group  $H'$  over  $K'$  and the action of  $H$  on  $X$  extends naturally to an action of  $H'$  on  $X'$ .  $X'$  and  $H'$  are  $K'$ -varieties, i.e. schemes of finite type over  $K'$ , which are defined over  $K$ , as well as the action  $H' \times X' \rightarrow X'$  (in the sense of Borel, cf. [Bor 91], i.e. given by polynomial data, where the polynomials have coefficients in  $K$ , with points being closed points). For  $x \in X$  (i.e. a  $K$ -rational point of  $X'$ ) the orbit  $H'x$  is a subvariety of  $X'$  defined over  $K$  and the orbit map  $\phi' : H' \rightarrow X', h \mapsto hx$ , is also defined over  $K$ .

(2) Recall that for  $X$  an algebraic  $K$ -variety with structure sheaf  $\mathcal{O}_X$ , the Zariski-tangent space  $T_x X$  of  $X$  at the point  $x \in X$  can be described as

$$T_x X := \{ K\text{-algebra homomorphisms } f : \mathcal{O}_{X,x} \rightarrow K[\epsilon] \mid p \circ f = \chi_x \},$$

where  $\mathcal{O}_{X,x}$  is the local ring of  $X$  at  $x$ ,  $K[\epsilon]$  the ring of dual numbers  $K[\epsilon] = K[t]/\langle t^2 \rangle = \{a + b\epsilon \mid a, b \in K, \epsilon^2 = 0\}$ , and  $p$  and  $\chi_x$  are the canonical residue maps  $K[\epsilon] \rightarrow K[\epsilon]/\langle \epsilon \rangle = K$  and  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x = K$ , where  $\mathfrak{m}_x$  is the maximal ideal of  $\mathcal{O}_{X,x}$ .

In the same way we have the tangent space  $T_x X'$  defined w.r.t.  $K'$ , satisfying

$$T_x X' = T_x X \otimes_K K'.$$

(3) The tangent map  $To' : T_e H' \rightarrow T_x X'$ ,  $e \in H$  the identity element, of the orbit map  $o'$  is defined over  $K$  (induced by  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{H,e}$ ) and hence induces a map  $T_e H \rightarrow T_x X$  which we denote by  $To$ , satisfying  $To' = To \otimes_K K'$ . We define

$$\begin{aligned} \tilde{T}_x(Hx) : &= \text{im}(To : T_e H \rightarrow T_x X), \\ \tilde{T}_x(H'x) : &= \text{im}(To' : T_e H' \rightarrow T_x X') \end{aligned}$$

and call it the *tangent image* at  $x$  of the orbit map. Obviously we have

$$\tilde{T}_x(H'x) = \tilde{T}_x(Hx) \otimes_K K'.$$

The following proposition is well-known (cf. [FSR05] Theorem 3.1).

**Proposition 2.2.** *Let  $K'$  be an algebraically closed field,  $X'$  an algebraic  $K'$ -variety and  $H'$  an algebraic group over  $K'$  acting algebraically on  $X'$ . For  $x \in X'$  the orbit map  $o' : H' \rightarrow H'x$  is separable iff the tangent map  $To' : T_e H' \rightarrow T_x(H'x)$  is surjective.*

Recall that the orbit map is called separable if the extension of fields of rational functions  $K(H'x) \subset K(H')$  is separably generated, i.e. there is a transcendence base  $\{x_i\}$  of  $K(H'x) \subset K(H')$  such that  $K(H')$  is a separable algebraic extension of  $K(H'x)(\{x_i\})$ . Separability holds e.g. if  $\text{char}(K') = 0$ .

**Corollary 2.3.** *Let  $K$  be a field,  $X$  an algebraic  $K$ -variety and  $H$  an algebraic group over  $K$  acting algebraically on  $X$ . Let  $K'$  be an algebraically closed extension field of  $K$  and let  $X'$  and  $H'$  be as in Remark 2.1. For  $x \in X$  the following are equivalent:*

- (i) *The orbit map  $o' : H' \rightarrow H'x$  is separable.*
- (ii)  $\dim_K \tilde{T}_x(Hx) = \dim_x H'x$
- (iii)  $\tilde{T}_x(Hx) = T_x(Hx)$

These conditions hold in particular if  $\text{char}(K) = 0$ .

*Proof.* Since the orbit  $H'x$  is smooth we have  $\dim_x H'x = \dim_{K'} T_x(H'x)$  and the latter equals  $\dim_K T_x(Hx)$  by Remark 2.1. This shows the equivalence of (ii) and (iii). The equality in (iii) is equivalent to  $\tilde{T}_x(H'x) = T_x(H'x)$  and hence (iii) is equivalent to (i) by Proposition 2.2.  $\square$

The following easy lemma is used below; it replaces the Taylor series in positive characteristic.

**Lemma 2.4.** *Let  $f(\mathbf{x}) = \sum_{|\alpha| \geq \text{ord}(f)} c_\alpha \mathbf{x}^\alpha \in K[[\mathbf{x}]]$  and  $\mathbf{z} = (z_1, \dots, z_s)$  new variables. Then*

$$f(\mathbf{x} + \mathbf{z}) = f(\mathbf{x}) + \sum_{\nu=1}^s \frac{\partial f(\mathbf{x})}{\partial x_\nu} \cdot z_\nu + \sum_{|\alpha| \geq \text{ord}(f)} c_\alpha \cdot \left( \sum_{\substack{|\gamma| \geq 2 \\ \gamma \leq \alpha}} \binom{\alpha_1}{\gamma_1} \cdot \dots \cdot \binom{\alpha_s}{\gamma_s} \mathbf{x}^{\alpha-\gamma} \mathbf{z}^\gamma \right),$$

where  $\gamma \leq \alpha$  means that  $\gamma_\nu \leq \alpha_\nu$  for all  $\nu = 1, \dots, s$ ,  $\binom{\alpha_\nu}{\gamma_\nu} \in \mathbb{Z}$  for all  $\nu = 1, \dots, s$  and if  $\text{char}(K) = p > 0$  we denote for  $k \in \mathbb{Z}$

$$k \mathbf{x}^{\alpha-\gamma} \mathbf{z}^\gamma = \begin{cases} 0 & \text{if } p \mid k \\ k(\text{mod } p) \mathbf{x}^{\alpha-\gamma} \mathbf{z}^\gamma & \text{if } p \nmid k \end{cases}$$

Denote by

$$E_{m,pq} \text{ (resp. } E_{n,hl})$$

the  $(p, q)$ -th (resp.  $(h, l)$ -th) canonical matrix of the ring of square matrices  $M_m := \text{Mat}(m, m, R)$  (resp.  $M_n := \text{Mat}(n, n, R)$ ) with 1 at the place  $(p, q)$  (resp.  $(h, l)$ ) and 0 else. For  $A = [a_{ij}(\mathbf{x})] \in M_{m,n}$  we set  $\frac{\partial A}{\partial x_\nu} = \left[ \frac{\partial a_{ij}(\mathbf{x})}{\partial x_\nu} \right] \in M_{m,n}$ .

**Proposition 2.5.** *Let  $A \in M_{m,n}$  and  $k \geq 1$ . Let  $G = \mathcal{G}_{lr}$  and let  $e = (I_m, I_n, \text{id}_R) \in G^{(k)}$  be the identity of the group  $G^{(k)}$ . Then the tangent image, i.e. the image of the tangent map*

$$T_e G^{(k)} \rightarrow T_{\text{jet}_k(A)} \left( G^{(k)} \text{jet}_k(A) \right),$$

considered as a subspace of  $M_{m,n}^{(k)}$ , is the submodule

$$\begin{aligned} & \tilde{T}_{\text{jet}_k(A)} \left( G^{(k)} \text{jet}_k(A) \right) := \\ & \left( \langle E_{m,pq} \cdot A \rangle + \langle A \cdot E_{n,hl} \rangle + \mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle + \mathfrak{m}^{k+1} \cdot M_{m,n} \right) / \mathfrak{m}^{k+1} \cdot M_{m,n}, \end{aligned}$$

where  $\langle E_{m,pq} \cdot A \rangle$ ,  $\langle A \cdot E_{n,hl} \rangle$ , and  $\left\langle \frac{\partial A}{\partial x_\nu} \right\rangle$  are the  $R$ -submodules of  $M_{m,n}$  generated by  $E_{m,pq} \cdot A$ ,  $p, q = 1, \dots, m$ ,  $A \cdot E_{n,hl}$ ,  $h, l = 1, \dots, n$ , and  $\frac{\partial A}{\partial x_\nu}$ ,  $\nu = 1, \dots, s$ , respectively.

Moreover, the tangent images of  $\mathcal{R}$ ,  $\mathcal{G}_l$ , and  $\mathcal{G}_r$  are respectively

$$\begin{aligned}\tilde{T}_{jet_k(A)} \left( \mathcal{R}^{(k)} jet_k(A) \right) &:= \left( \mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle + \mathfrak{m}^{k+1} \cdot M_{m,n} \right) / \mathfrak{m}^{k+1} \cdot M_{m,n}, \\ \tilde{T}_{jet_k(A)} \left( \mathcal{G}_l^{(k)} jet_k(A) \right) &:= \left( \langle E_{m,pq} \cdot A \rangle + \mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle + \mathfrak{m}^{k+1} \cdot M_{m,n} \right) / \mathfrak{m}^{k+1} \cdot M_{m,n}, \\ \tilde{T}_{jet_k(A)} \left( \mathcal{G}_r^{(k)} jet_k(A) \right) &:= \left( \langle A \cdot E_{n,hl} \rangle + \mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle + \mathfrak{m}^{k+1} \cdot M_{m,n} \right) / \mathfrak{m}^{k+1} \cdot M_{m,n}.\end{aligned}$$

*Proof.* The orbit map

$$\begin{aligned}o^{(k)} : G^{(k)} &\rightarrow G^{(k)} jet_k(A), \\ (jet_k(U), jet_k(V), jet_k(\phi)) &\mapsto jet_k(U \cdot \phi(A) \cdot V) = jet_k(U \cdot [a_{ij}(\phi(\mathbf{x}))] \cdot V),\end{aligned}$$

where  $A = [a_{ij}(\mathbf{x})]$ , induces the tangent map

$$T_e G^{(k)} \rightarrow T_{jet_k(A)} \left( G^{(k)} jet_k(A) \right).$$

Each element of  $T_e G^{(k)}$  can be represented by a triple

$$(jet_k(I_n + \epsilon \cdot U), jet_k(I_m + \epsilon \cdot V), jet_k(id_R + \epsilon \cdot \phi)),$$

where  $U \in M_m$ ,  $V \in M_n$ , and  $\phi(x_\nu) =: \phi_\nu \in \mathfrak{m}$  for all  $\nu = 1, \dots, s$ . Letting this triple act on  $jet_k(A)$  we get

$$jet_k((I_m + \epsilon \cdot U) \cdot [a_{ij}(\mathbf{x} + \epsilon \cdot \phi(\mathbf{x}))] \cdot (I_n + \epsilon \cdot V)),$$

where  $\phi(\mathbf{x}) := (\phi(x_1), \dots, \phi(x_s)) = (\phi_1, \dots, \phi_s)$ .

Now for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , let  $o_{ij} := \text{ord}(a_{ij})$  and apply Lemma 2.4 to

$$f = a_{ij} = \sum_{|\alpha| \geq o_{ij}} c_\alpha^{(ij)} \mathbf{x}^\alpha, \quad \mathbf{z} = \epsilon \cdot \phi(\mathbf{x}).$$

Since  $\epsilon^2 = 0$  we have

$$a_{ij}(\mathbf{x} + \mathbf{z}) = a_{ij}(\mathbf{x}) + \epsilon \cdot \sum_{\nu=1}^s \frac{\partial a_{ij}(\mathbf{x})}{\partial x_\nu} \cdot \phi_\nu.$$

This implies

$$(I_m + \epsilon \cdot U) \cdot [a_{ij}(\mathbf{x} + \epsilon \cdot \phi(\mathbf{x}))] \cdot (I_n + \epsilon \cdot V) = A + \epsilon \cdot \left( A \cdot V + U \cdot A + \sum_{\nu=1}^s \phi_\nu \cdot \frac{\partial A}{\partial x_\nu} \right)$$

so that the image of the triple under the tangent map is the  $k$ -jet of the matrix

$$A + \epsilon \cdot \left( A \cdot V + U \cdot A + \sum_{\nu=1}^s \phi_\nu \cdot \frac{\partial A}{\partial x_\nu} \right).$$

Hence, the claim follows for  $\mathcal{G}_{lr}$ . The proofs for the other groups are similar.  $\square$

From now on, for  $A \in M_{m,n}$ , we use the notations  $\langle E_{m,pq} \cdot A \rangle$ ,  $\langle A \cdot E_{n,hl} \rangle$ , and  $\left\langle \frac{\partial A}{\partial x_\nu} \right\rangle$  as in Proposition 2.5.

**Definition 2.6.** For  $A \in M_{m,n}$ , we call the  $R$ -submodules of  $M_{m,n}$

$$\begin{aligned} \tilde{T}_A(\mathcal{R}A) &:= \mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle, \\ \tilde{T}_A(\mathcal{G}_l A) &:= \langle E_{m,pq} \cdot A \rangle + \mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle, \\ \tilde{T}_A(\mathcal{G}_r A) &:= \langle A \cdot E_{n,hl} \rangle + \mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle, \text{ and} \\ \tilde{T}_A(\mathcal{G}_{lr} A) &:= \langle E_{m,pq} \cdot A \rangle + \langle A \cdot E_{n,hl} \rangle + \mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle \end{aligned}$$

the **tangent images** at  $A$  to the orbit of  $A$  under the actions of  $\mathcal{R}$ ,  $\mathcal{G}_l$ ,  $\mathcal{G}_r$ , and  $\mathcal{G}_{lr}$  on  $M_{m,n}$ , respectively.

Replacing  $\mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle$  by  $R \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle$  in the above definition we get the **extended tangent images**  $\tilde{T}_A^e(\mathcal{R}A)$ ,  $\tilde{T}_A^e(\mathcal{G}_l A)$ ,  $\tilde{T}_A^e(\mathcal{G}_r A)$ , and  $\tilde{T}_A^e(\mathcal{G}_{lr} A)$ .

Note that  $\{\tilde{T}_{jet_k(A)}(G^{(k)} jet_k(A)), \pi_k\}_k \subset M_{m,n}^{(k)}$  is an inverse system of  $R$ -modules, where  $\pi_k$  is induced by the canonical projection  $M_{m,n}^{(k)} \rightarrow M_{m,n}^{(k-1)}$ , and we have

$$\tilde{T}_A(GA) = \varprojlim_{k \geq 0} \tilde{T}_{jet_k(A)}(G^{(k)} jet_k(A)) \subset M_{m,n}.$$

Likewise,  $\{T_{jet_k(A)}(G^{(k)} jet_k(A)), \pi_k\}_k \subset M_{m,n}^{(k)}$  is an inverse system of  $K$ -vector spaces and we call the  $K$ -vector space

$$T_A(GA) := \varprojlim_{k \geq 0} T_{jet_k(A)}(G^{(k)} jet_k(A)) \subset M_{m,n}.$$

the **tangent space** at  $A$  to the orbit  $GA$ .



**Remark 2.7.** We have inclusions  $\tilde{T}_{jet_k(A)}(G^{(k)}jet_k(A)) \subset T_{jet_k(A)}(G^{(k)}jet_k(A))$  for the  $k$ -jets and hence  $\tilde{T}_A(GA) \subset T_A(GA)$  with equality if the  $k$ -jets coincide for sufficiently big  $k$ . Below we give an example with  $\tilde{T}_A(GA) \neq T_A(GA)$ .

Using Corollary 2.3 we get

**Lemma 2.8.**  $\tilde{T}_A(GA) = T_A(GA)$  if there is an integer  $k$  such that the orbit map  $o^{(l)} : G^{(l)} \rightarrow G^{(l)}jet_l(A)$  (over an algebraic closure of  $K$ ) is separable for all  $l \geq k$  (e.g. if  $\text{char}(K) = 0$ ). Conversely,  $\tilde{T}_A(GA) = T_A(GA)$  implies that  $o^{(l)}$  is separable for all  $l$ .

In the following, when we say that the orbit map is separable, we mean that this holds over an algebraic closure of  $K$ . Also the dimension of a  $K$ -variety is its dimension over an algebraic closure.

**Example 2.9.** We give examples where the tangent image is strictly contained in the tangent space, i.e. the tangent map

$$T_e G^{(k)} \rightarrow T_{jet_k(A)}(G^{(k)}jet_k(A))$$

is not surjective and hence the orbit map  $G^{(k)} \rightarrow G^{(k)}jet_k(f)$  is not separable.

1. Let  $\text{char}(K) = p > 0$  and  $k \geq p$ . Let the right group  $G = \mathcal{R}$  act on  $K[[x_1, \dots, x_s]]$  and let  $f = x_1^p + \dots + x_s^p$ . Then  $j(f) = 0$  so that the tangent image  $\mathfrak{m} \cdot j(f)$  is zero. On the other hand, for  $\phi \in \mathcal{R}$ ,  $\phi = (ax_1, x_2, \dots, x_s)$  where  $a \neq 0$  and  $a^p - 1 \neq 0$ , we have

$$jet_k(f \circ \phi) = a^p x_1^p + x_2^p + \dots + x_s^p \neq x_1^p + x_2^p + \dots + x_s^p = jet_k(f),$$

showing that the orbit of  $jet_k(f)$  and hence its tangent space has dimension  $\geq 1$ . Note that  $f$  is not finitely  $\mathcal{R}$ -determined.

2. Let  $\text{char}(K) = 2$ ,  $f = x^2 + y^3$  and let the contact group  $G = \mathcal{G}_l$  act on  $R = K[[x, y]]$ . We compute:

- $f$  is  $\mathcal{G}_l$  4-determined
- the tangent image  $\tilde{T}_f(\mathcal{G}_l f) = \langle f \rangle + \mathfrak{m} \cdot j(f)$  is  $\langle x^2, xy^2, y^3 \rangle$
- its 4-jet  $\tilde{T}_{jet_4(f)}(\mathcal{G}_l^{(4)} jet_4(f))$  has dimension 10 in  $R^{(4)} = R/\mathfrak{m}^5$
- the group  $\mathcal{G}_l^{(4)}$  has dimension 43 and the stabilizer of  $jet_4(f)$  has dimension 32 in  $R^{(4)}$ .

It follows that the orbit  $\mathcal{G}_l^{(4)} jet_4(f)$  has dimension 11 in  $R^{(4)}$ . Since the tangent image has dimension 10,  $\tilde{T}_f(\mathcal{G}_l f) \neq T_f(\mathcal{G}_l f)$  by Corollary 2.3.

3. Similar examples exist in other positive characteristics. E.g. we have in characteristic 3 for  $f = x^3 + y^4$  that the dimension in  $R^{(5)}$  of the  $\mathcal{G}_l$ -tangent image is 11 while that of the tangent space is 12.

On the other hand we can show that for  $f = x^2 + y^3$  and  $G = \mathcal{G}_l$  the tangent image coincides with the tangent space in characteristic 3 and for  $G = \mathcal{R}$  also in characteristic 2, implying that in these cases the orbit map is separable.

The computations were done by using SINGULAR (cf. [DGPS15]). The determinacy (using Theorem 3.2) and the tangent image of the orbit map are easily computable by using standard bases in the local ring  $R$  (see [GP08]). To compute the tangent space to the orbit we need to compute equations of the orbit, which is possible but much harder, as in general a large number of group variables has to be eliminated. The dimension of the orbit is usually easier obtained by computing the stabilizer. Note that the dimension of a variety computed by standard resp. Gröbner bases refers to the dimension over the algebraic closure. Details of the computational part will appear elsewhere.

**Remark 2.10.** The examples above show that the statement of Proposition 1 in [BGM12] is wrong in positive characteristic for the right group as well as for the contact group, for the latter even if  $f$  is finitely determined. Hence the proof of the “if” part of Theorem 5 in [BGM12] contains a gap. However, this gap can be closed as we show in our paper [GP16] by proving a more general theorem for complete intersections.

We finish this section with an obvious necessary condition for finite  $G$ -determinacy:

**Lemma 2.11.** *If  $A$  is finitely  $G$ -determined then*

$$\dim_K M_{m,n}/T_A(GA) < \infty.$$

Indeed, let  $A$  be  $k$ -determined,  $l > k$ , and  $B \in \mathfrak{m}^l \cdot M_{m,n}$ . Then  $A + tB \in GA$  for all  $t \in K$ . This yields  $\text{jet}_l(B) \in T_{\text{jet}_l(A)}(G^{(l)}\text{jet}_l(A))$  so that  $B \in T_A(GA)$ . This means that  $\mathfrak{m}^l \cdot M_{m,n} \subset T_A(GA)$ , and thus the claim follows.

### 3 A sufficient condition in terms of the tangent image

In this section we establish a sufficient condition for finite  $G$ -determinacy of matrices in terms of the tangent image defined in section 2.

**Lemma 3.1.** *Let  $A, B \in M_{m,n}$ .*

1. *If  $A \stackrel{\mathcal{R}}{\sim} B$ , i.e. there is an automorphism  $\phi \in \text{Aut}(R)$  such that  $B = \phi(A)$ , then, for the submodules of  $M_{m,n}$  generated by the partials, we have*

$$\tilde{T}_B^e(\mathcal{R}B) = \phi\left(\tilde{T}_A^e(\mathcal{R}A)\right).$$

2. If  $A \stackrel{\mathcal{G}_l}{\sim} B$ , i.e. there are invertible matrices  $U \in GL(m, R)$ ,  $V \in GL(n, R)$ , and an automorphism  $\phi \in \text{Aut}(R)$  such that  $B = U \cdot \phi(A) \cdot V$ , then

$$\tilde{T}_B^e(\mathcal{G}_{lr}B) = U \cdot \phi \left( \tilde{T}_A^e(\mathcal{G}_{lr}A) \right) \cdot V.$$

The same holds for  $\tilde{T}_B(\mathcal{G}_{lr}B)$  and  $\tilde{T}_A(\mathcal{G}_{lr}A)$  instead of  $\tilde{T}_B^e(\mathcal{G}_{lr}B)$  and  $\tilde{T}_A^e(\mathcal{G}_{lr}A)$ . A similar result holds for  $\mathcal{G}_l$  and  $\mathcal{G}_r$ .

*Proof.* The proof of 1. is an application of the chain rule to the entries of  $B$  and for 2. use in addition the product rule.  $\square$

Our main result is the following theorem.

**Theorem 3.2.** Let  $A = [a_{ij}] \in \mathfrak{m} \cdot M_{m,n}$  and let  $o := \text{ord}(A)$ . If there is some integer  $k \geq 0$  such that

$$\mathfrak{m}^{k+2} \cdot M_{m,n} \subset \mathfrak{m} \cdot \tilde{T}_A(GA), \quad (1)$$

then  $A$  is  $G(2k - o + 2)$ -determined.

*Proof.* If  $A$  is the zero matrix, it is not finitely  $G$ -determined and the statement is true. Hence, we may assume that  $A \neq 0$ . Assume that

$$\mathfrak{m}^{k+2} \cdot M_{m,n} \subset \mathfrak{m} \cdot \langle E_{m,pq} \cdot A \rangle + \mathfrak{m} \cdot \langle A \cdot E_{n,hl} \rangle + \mathfrak{m}^2 \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle.$$

For all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , set  $o_{ij} := \text{ord}(a_{ij})$ . Choose  $h \in \{1, \dots, m\}$  and  $l \in \{1, \dots, n\}$  such that  $o = \text{ord}(a_{hl})$ . Looking at  $E_{hl}$  in  $M_{m,n}$ , the hypothesis implies that

$$\mathfrak{m}^{k+2} \subset \mathfrak{m} \cdot \langle a_{1l}, a_{2l}, \dots, a_{ml}, a_{h1}, a_{h2}, \dots, a_{hn} \rangle + \mathfrak{m}^2 \cdot \mathfrak{j}(a_{hl}) \subset \mathfrak{m}^{o+1}.$$

This yields  $k \geq o - 1$ . Set  $N := 2k - o + 2 \geq k + 1$ . Let  $B \in M_{m,n}$  be such that  $B - A \in \mathfrak{m}^{N+1} \cdot M_{m,n}$ . We show that  $B \stackrel{G}{\sim} A$ . To prove this, we will construct inductively sequences of matrices  $\{X_t\}_{t \geq 0} \subset GL(m, R)$ ,  $\{Y_t\}_{t \geq 0} \subset GL(n, R)$ , and a sequence of automorphisms  $\{\varphi_t\}_{t \geq 1} \subset \text{Aut}(R)$  such that  $\{X_t \cdot \varphi_t(A) \cdot Y_t\}_{t \geq 1}$  converges in the  $\mathfrak{m}$ -adic topology to  $X \cdot \varphi(A) \cdot Y$  for some  $X \in GL(m, R)$ ,  $Y \in GL(n, R)$ , and  $\varphi \in \text{Aut}(R)$ , and such that

$$B - X_t \cdot \varphi_t(A) \cdot Y_t \in \mathfrak{m}^{N+1+t} \cdot M_{m,n}$$

holds for all  $t \geq 1$ . Then we obtain  $B = X \cdot \varphi(A) \cdot Y$ .

For this, we first construct sequences of matrices  $\{U_t\}_{t \geq 1} \subset GL(m, R)$ ,  $\{V_t\}_{t \geq 1} \subset GL(n, R)$ , and  $\{A_t\}_{t \geq 0} \subset M_{m,n}$  with  $A_0 = A$  and a sequence of automorphisms  $\{\phi_t\}_{t \geq 1} \subset$

$\text{Aut}(R)$  such that for all  $t \geq 1$ , we have

i)  $A_t = U_t \cdot \phi_t(A_{t-1}) \cdot V_t$

ii)  $B - A_t \in \mathfrak{m}^{N+t+1} \cdot M_{m,n}$ .

Setting  $Q := N - k \geq 1$  we have by assumption

$$B - A \in \mathfrak{m}^{N+1} \cdot M_{m,n} = \mathfrak{m}^{Q-1} \cdot \mathfrak{m}^{k+2} \cdot M_{m,n} \subset \mathfrak{m}^Q \cdot \langle E_{m,pq} \cdot A \rangle + \mathfrak{m}^Q \cdot \langle A \cdot E_{n,hl} \rangle + \mathfrak{m}^{Q+1} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle.$$

Hence, there are  $u_{pq}^{(1)} \in \mathfrak{m}^Q$ ,  $v_{hl}^{(1)} \in \mathfrak{m}^Q$ , and  $d_{1,\nu} \in \mathfrak{m}^{Q+1}$  for all  $p, q = 1, \dots, m$ ,  $h, l = 1, \dots, n$ , and  $\nu = 1, \dots, s$  such that

$$\begin{aligned} B - A &= \sum_{p,q=1}^m u_{pq}^{(1)} \cdot E_{m,pq} \cdot A + \sum_{h,l=1}^n v_{hl}^{(1)} \cdot A \cdot E_{n,hl} + \sum_{\nu=1}^s d_{1,\nu} \cdot \frac{\partial A}{\partial x_\nu} \\ &= U^{(1)} \cdot A + A \cdot V^{(1)} + \sum_{\nu=1}^s d_{1,\nu} \cdot \frac{\partial A}{\partial x_\nu}, \end{aligned}$$

where  $U^{(1)} := [u_{pq}^{(1)}] \in \mathfrak{m}^Q \cdot M_m$  and  $V^{(1)} := [v_{hl}^{(1)}] \in \mathfrak{m}^Q \cdot M_n$ .

Define  $U_1 := I_m + U^{(1)} \in GL(m, R)$ ,  $V_1 := I_n + V^{(1)} \in GL(n, R)$ ,

$$\phi_1 : R \rightarrow R, \quad x_\nu \mapsto x_\nu + d_{1,\nu}, \quad \nu = 1, \dots, s,$$

and  $A_1 := U_1 \cdot \phi_1(A) \cdot V_1$ . We now show that

$$B - A_1 \in \mathfrak{m}^{N+2} \cdot M_{m,n}.$$

For all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , applying Lemma 2.4 to  $f = a_{ij} = \sum_{|\alpha| \geq o_{ij}} c_\alpha^{(ij)} \mathbf{x}^\alpha$  and

$(z_1, \dots, z_s) = (d_{1,1}, \dots, d_{1,s})$  we have

$$\phi_1(a_{ij}(\mathbf{x})) = a_{ij}(x_1 + d_{1,1}, \dots, x_s + d_{1,s}) = a_{ij}(\mathbf{x}) + \sum_{\nu=1}^s \frac{\partial a_{ij}(\mathbf{x})}{\partial x_\nu} \cdot d_{1,\nu} + h_{ij},$$

where

$$h_{ij} = \sum_{|\alpha| \geq o_{ij}} c_\alpha^{(ij)} \cdot \left( \sum_{\substack{|\gamma| \geq 2 \\ \gamma \leq \alpha}} \binom{\alpha_1}{\gamma_1} \cdot \dots \cdot \binom{\alpha_s}{\gamma_s} \mathbf{x}^{\alpha-\gamma} \cdot d_{1,1}^{\gamma_1} \cdot \dots \cdot d_{1,s}^{\gamma_s} \right).$$

Then for all  $i$  and  $j$ , for  $\alpha \in \mathbb{N}^s$ ,  $|\alpha| \geq o_{ij}$ , and for  $\gamma \in \mathbb{N}^s$ ,  $\gamma \leq \alpha$ ,  $|\gamma| \geq 2$ , we have

$$|\alpha| - |\gamma| + (Q+1)|\gamma| = |\gamma|Q + |\alpha| \geq 2Q + o_{ij} \geq 2Q + o = N + 2.$$

Thus, for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , we have  $h_{ij} \in \mathfrak{m}^{N+2}$ . We obtain

$$\phi_1(A) = A + \sum_{\nu=1}^s d_{1,\nu} \cdot \frac{\partial A}{\partial x_\nu} + H,$$

where  $H = [h_{ij}]$  and  $H \in \mathfrak{m}^{N+2} \cdot M_{m,n}$ . This implies that

$$\begin{aligned} B - A_1 &= B - \left( I_m + U^{(1)} \right) \cdot \phi_1(A) \cdot \left( I_n + V^{(1)} \right) = -U^{(1)} \cdot A \cdot V^{(1)} - U_1 \cdot H \cdot V_1 - \\ &\quad - \sum_{\nu=1}^s d_{1,\nu} \cdot U^{(1)} \cdot \frac{\partial A}{\partial x_\nu} - \sum_{\nu=1}^s d_{1,\nu} \cdot \frac{\partial A}{\partial x_\nu} \cdot V^{(1)} - \sum_{\nu=1}^s d_{1,\nu} \cdot U^{(1)} \cdot \frac{\partial A}{\partial x_\nu} \cdot V^{(1)} \\ &\in \mathfrak{m}^{N+2} \cdot M_{m,n}. \end{aligned}$$

Now that by assumption and Lemma 3.1

$$\mathfrak{m}^{k+2} \cdot M_{m,n} \subset \mathfrak{m} \cdot \langle E_{m,pq} \cdot A_1 \rangle + \mathfrak{m} \cdot \langle A_1 \cdot E_{n,hl} \rangle + \mathfrak{m}^2 \cdot \left\langle \frac{\partial(A_1)}{\partial x_\nu} \right\rangle.$$

Furthermore, since  $\text{ord}(A_1) = \text{ord}(A) = o$ , we can proceed inductively to construct the sequences  $\{A_t\}_{t \geq 0}$ ,  $\{U_t\}_{t \geq 1}$ ,  $\{V_t\}_{t \geq 1}$ , and  $\{\phi_t\}_{t \geq 1}$  as desired.

Now, for  $t \geq 1$ , we define

$$\begin{aligned} \varphi_t &:= \phi_t \circ \dots \circ \phi_1, \\ X_t &= U_t \cdot \phi_t(X_{t-1}), \quad X_0 = I_m, \text{ and} \\ Y_t &= \phi_t(Y_{t-1}) \cdot V_t, \quad Y_0 = I_n. \end{aligned}$$

Then by induction we obtain  $A_t = X_t \cdot \varphi_t(A) \cdot Y_t$  and  $B - X_t \cdot \varphi_t(A) \cdot Y_t \in \mathfrak{m}^{N+t+1} \cdot M_{m,n}$ .

It remains to prove that  $\{X_t \cdot \varphi_t(A) \cdot Y_t\}_{t \geq 1}$  converges to  $X \cdot \varphi(A) \cdot Y$  in the  $\mathfrak{m}$ -adic topology for some  $\varphi \in \text{Aut}(R)$ ,  $X \in GL(m, \bar{R})$ , and  $Y \in GL(n, R)$ . For that, we show that the sequences  $\{X_t\}_{t \geq 0}$ ,  $\{Y_t\}_{t \geq 0}$ , and  $\{\varphi_t(x_\nu)\}_{t \geq 1}$  for all  $\nu \in \{1, \dots, s\}$  are Cauchy sequences and define  $X$ ,  $Y$ , and  $\varphi$  as their limits, respectively. We have for all  $t \geq 1$ ,

$$X_t - X_{t-1} = \left( I_m + U^{(t)} \right) \cdot \phi_t(X_{t-1}) - X_{t-1} = \phi_t(X_{t-1}) - X_{t-1} + U^{(t)} \cdot \phi_t(X_{t-1}) \in \mathfrak{m}^{Q+t-1} \cdot M_m$$

since  $\phi_t(X_{t-1}) - X_{t-1} \in \mathfrak{m}^{Q+t} \cdot M_m$  and  $U^{(t)} \in \mathfrak{m}^{Q+t-1} \cdot M_m$ .

Hence, given  $P \geq 1$ , for  $t > r \geq r_0$  with  $r_0 = \max\{P - Q, 1\}$  we have

$$X_t - X_r = (X_t - X_{t-1}) + \dots + (X_{r+1} - X_r) \in \mathfrak{m}^{Q+r} \cdot M_m \subset \mathfrak{m}^P \cdot M_m.$$

This shows that  $\{X_t\}_{t \geq 0}$  is a Cauchy sequence in  $M_m$ , and thus it converges to a matrix  $X \in M_m$ . Moreover, it follows by induction that

$$X_t - I_m \in \mathfrak{m}^Q \cdot M_m$$

so that  $X = I_m + X_0$  for some  $X_0 \in \mathfrak{m}^Q \cdot M_m$ . By the same argument,  $\{Y_t\}_{t \geq 0}$  converges to a  $Y \in GL(n, R)$ . Now fix  $\nu \in \{1, \dots, s\}$  and express  $\phi_t(\varphi_{t-1}(x_\nu))$  in term of  $\varphi_{t-1}(x_\nu)$  as above, we have

$$\varphi_t(x_\nu) - \varphi_{t-1}(x_\nu) = \phi_t(\varphi_{t-1}(x_\nu)) - \varphi_{t-1}(x_\nu) \in \mathfrak{m}^{Q+t}$$

since  $d_{t,\nu} \in \mathfrak{m}^{Q+t}$  for all  $\nu = 1, \dots, s$ . By a similar argument as above,  $\{\varphi_t(x_\nu)\}_{t \geq 1}$  is a Cauchy sequence and hence converges in  $R$ . In addition, by induction we have

$$\varphi_t(x_\nu) - x_\nu = \varphi_t(x_\nu) - \varphi_{t-1}(x_\nu) + \varphi_{t-1}(x_\nu) - x_\nu \in \mathfrak{m}^{Q+1}.$$

This implies that  $\{\varphi_t(x_\nu)\}_{t \geq 1}$  converges to a power series  $x_\nu + d_\nu$  for some  $d_\nu \in \mathfrak{m}^{Q+1}$ . Define the automorphism  $\varphi$  by

$$\varphi : R \rightarrow R, \quad x_\nu \mapsto x_\nu + d_\nu, \quad \nu = 1, \dots, s.$$

Then for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ ,

$$\varphi_t(a_{ij}) - \varphi(a_{ij}) \in \mathfrak{m}^P,$$

which implies  $\varphi_t(A) - \varphi(A) \in \mathfrak{m}^P \cdot M_{m,n}$ . On the other hand, since  $\{X_t\}_{t \geq 0}$  converges to  $X$  and  $\{Y_t\}_{t \geq 0}$  converges to  $Y$ , there are  $t_1$  and  $t_2 \in \mathbb{N}$  such that  $X_t - X \in \mathfrak{m}^P \cdot M_m$  for all  $t \geq t_1$  and  $Y_t - Y \in \mathfrak{m}^P \cdot M_n$  for all  $t \geq t_2$ . Hence, for  $t \geq t_3$  with  $t_3 := \max\{t_0, t_1, t_2\} \geq 1$ , we have

$$\begin{aligned} X_t \cdot \varphi_t(A) \cdot Y_t - X \cdot \varphi(A) \cdot Y &= X_t \cdot \varphi_t(A) \cdot (Y_t - Y) + X_t \cdot (\varphi_t(A) - \varphi(A)) \cdot Y \\ &\quad + (X_t - X) \cdot \varphi(A) \cdot Y \in \mathfrak{m}^P \cdot M_{m,n}. \end{aligned}$$

By uniqueness of the limit, we get  $B = X \cdot \varphi(A) \cdot Y$ . □

**Remark 3.3.** 1. For  $\text{char}(K) = 0$ , condition (1) is also necessary for finite determinacy, see Proposition 4.3. We do not know whether this is true in positive characteristic for arbitrary  $m, n$ . For  $n = 1$  and  $G = \mathcal{G}_{lr}$  this is true as shown in [GP16] (here  $\mathcal{G}_{lr}$ -equivalence coincides with  $\mathcal{G}_l$ -equivalence).

2. The determinacy bound given in Theorem 3.2 is new also in characteristic zero, but we expect that it can be improved by using integration of vector fields.

## 4 Finite determinacy for separable orbit maps

We provide a criterion saying that for a matrix  $A \in \mathfrak{m} \cdot M_{m,n}$ , if the orbit map  $G^{(k)} \rightarrow G^{(k)} \text{jet}_k(A)$  is separable for all  $k \geq k_0$ , some  $k_0 \in \mathbb{N}$ , then finite  $G$ -determinacy and finite codimension of the tangent image  $\tilde{T}_A(GA)$  are equivalent.

We derive first equivalent conditions to finite codimension of  $\tilde{T}_A(GA)$  for which we need the following result from commutative algebra.

**Lemma 4.1.** *Let  $(R, \mathfrak{m})$  be a local Noetherian  $K$ -algebra and  $L$  a finitely generated  $R$ -module. Then the following are equivalent:*

1.  $\dim_K L < \infty$ .
2.  $\mathfrak{m}^k \cdot L = 0$  for some  $k$ .
3. Let  $R^t \xrightarrow{\Theta} R^l \rightarrow L \rightarrow 0$  be a presentation of  $L$ , then there is some  $h$  such that  $\mathfrak{m}^h \subset I_l(\Theta)$ , where  $I_l(\Theta)$  is the  $l$ -th Fitting ideal of  $L$ , i.e. the ideal generated by all  $l \times l$  minors of the presentation matrix  $\Theta$ .

Moreover, with the same notation as in 3.,  $L = 0$  if and only if  $I_l(\Theta) = R$ .

**Supplement:** If  $\mathfrak{m}^h \subset I_l(\Theta)$  for some positive integer  $h$  then  $\mathfrak{m}^h \cdot L = 0$ .

*Proof.* Follows easily from [Eis95, Proposition 20.7]. □

**Proposition 4.2.** *Let  $A \in \mathfrak{m} \cdot M_{m,n}$ . Then  $A$  is finitely  $G$ -determined if one of the following equivalent statements holds:*

1.  $\mathfrak{m}^{l+1} \cdot M_{m,n} \subset \tilde{T}_A(GA)$  for some positive integer  $l$ .
2.  $\dim_K (\mathfrak{m} \cdot M_{m,n} / \tilde{T}_A(GA)) =: d < \infty$ .
3.  $\dim_K (M_{m,n} / \tilde{T}_A^e(GA)) =: d_e < \infty$ .
4.  $\mathfrak{m}^h \cdot M_{m,n} \subset \tilde{T}_A^e(GA)$  for some positive integer  $h$ .
5.  $\mathfrak{m}^k \subset I_{mn}(\Theta_{(G,A)})$  for some positive integer  $k$ , where

$$R^t \xrightarrow{\Theta_{(G,A)}} M_{m,n} \rightarrow M_{m,n} / \tilde{T}_A^e(GA) \rightarrow 0$$

is a presentation of  $M_{m,n} / \tilde{T}_A^e(GA)$ , i.e.  $V(I_{mn}(\Theta_{(G,A)})) = \{\mathfrak{m}\}$ .

6.  $\text{Supp}(M_{m,n} / \tilde{T}_A^e(GA)) = \{\mathfrak{m}\}$ , i.e.  $(M_{m,n})_P = (\tilde{T}_A^e(GA))_P$  for all  $P \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ .

Furthermore, if the condition 1. (resp. 2., 3., 4., and 5.) above holds then  $A$  is  $G$   $(2c - \text{ord}(A) + 2)$ -determined, where  $c = l$  (resp.  $d$ ,  $d_e$ ,  $h$ , and  $k$ ).

*Proof.* By Theorem 3.2 condition 1. implies finite determinacy. We prove that the six conditions are equivalent:

(1.  $\Rightarrow$  2.) This is clear since  $\dim_K (\mathfrak{m} \cdot M_{m,n} / \mathfrak{m}^{l+1} \cdot M_{m,n}) < \infty$ .

(2.  $\Rightarrow$  3.) Since  $\tilde{T}_A(GA) \subset \tilde{T}_A^e(GA)$ , we have

$$\dim_K (M_{m,n} / \tilde{T}_A^e(GA)) \leq \dim_K (M_{m,n} / \tilde{T}_A(GA)) < \infty.$$

Now use that the difference between  $\dim_K (M_{m,n} / \tilde{T}_A(GA))$  and  $\dim_K (\mathfrak{m} \cdot M_{m,n} / \tilde{T}_A(GA))$  is finite.

(3.  $\Rightarrow$  4.) We have a chain of  $K$ -vector subspaces:

$$M_{m,n} / \tilde{T}_A^e(GA) \supset (\mathfrak{m} \cdot M_{m,n} + \tilde{T}_A^e(GA)) / \tilde{T}_A^e(GA) \supset \dots$$

If  $\dim_K (M_{m,n} / \tilde{T}_A^e(GA)) = d_e$  then  $(\mathfrak{m}^{d_e} \cdot M_{m,n} + \tilde{T}_A^e(GA)) / \tilde{T}_A^e(GA) = 0$  so that  $\mathfrak{m}^{d_e} \cdot M_{m,n} \subset \tilde{T}_A^e(GA)$ .

(4.  $\Rightarrow$  1.) This is obvious.

(4.  $\Leftrightarrow$  5.) Apply Lemma 4.1 (2.  $\Leftrightarrow$  3.) to  $L = M_{m,n} / \tilde{T}_A^e(GA)$ .

(5.  $\Leftrightarrow$  6.). Let  $R^t \xrightarrow{\Theta_{(G,A)}} M_{m,n} \rightarrow M_{m,n} / \tilde{T}_A^e(GA) \rightarrow 0$  be a presentation of  $M_{m,n} / \tilde{T}_A^e(GA)$ . By [Eis95, Proposition 20.7],

$$\sqrt{I_{mn}(\Theta_{(G,A)})} = \sqrt{\text{Ann}_R(M_{m,n} / \tilde{T}_A^e(GA))}.$$

This implies the zero sets of the two ideals  $I_{mn}(\Theta_{(G,A)})$  and  $\text{Ann}_R(M_{m,n} / \tilde{T}_A^e(GA))$  define the same varieties in  $\text{Spec}(R)$ . That means

$$\text{Supp}(M_{m,n} / \tilde{T}_A^e(GA)) = V(I_{mn}(\Theta_{(G,A)}))$$

and thus the two assertions are equivalent.

To derive the determinacy bound, we apply Theorem 3.2. By multiplying the inclusion in 1. (resp. 4) with  $\mathfrak{m}$  (resp.  $\mathfrak{m}^2$ ) and applying Theorem 3.2, we get the bound. Assume that the condition in 3. (resp. 2.) holds. Using the argument as in the proof of (3.  $\Leftrightarrow$  4.) we obtain that  $\mathfrak{m}^{d_e} \cdot M_{m,n} \subset \tilde{T}_A^e(GA)$  (resp.  $\mathfrak{m}^{d+1} \cdot M_{m,n} \subset \tilde{T}_A(GA)$ ). Applying the determinacy bound for 4. (resp. 1.), we get the claim. Now assume that condition 5. holds. Apply the supplement of Lemma 4.1 to  $L = M_{m,n} / \tilde{T}_A^e(GA)$ , we have  $\mathfrak{m}^k \cdot L = 0$ . This implies  $\mathfrak{m}^k \cdot M_{m,n} \subset \tilde{T}_A^e(GA)$ , and thus  $A$  is  $G$   $(2k - \text{ord}(A) + 2)$ -determined.  $\square$



**Theorem 4.3.** *Let  $A \in \mathfrak{m} \cdot M_{m,n}$ .*

1. *Assume that  $A$  is finitely  $G$ -determined and the orbit map  $G^{(l)} \rightarrow G^{(l)} \text{jet}_l(A)$  is separable for some  $l > k$ , where  $k$  is a  $G$ -determinacy bound of  $A$ . Then the equivalent conditions in Proposition 4.2 hold.*
2. *Assume that there is some  $k \in \mathbb{N}$  such that the orbit map  $G^{(l)} \rightarrow G^{(l)} \text{jet}_l(A)$  is separable for all  $l \geq k$  (e.g. if  $\text{char}(K) = 0$ ). Then  $A$  is finitely  $G$ -determined if and only if one of the equivalent conditions in Proposition 4.2 holds.*

The criterion of the following corollary was proved in [Mat68] for complex or real analytic and  $\mathbb{C}^\infty$  map germs, i.e.  $M_{m,1}$ . It follows for arbitrary matrices from Theorem 4.3, Theorem 3.2 and Lemma 2.8 (and was stated in [BK16, 1.4] without proof):

**Corollary 4.4.** *If  $\text{char}(K) = 0$  then the matrix  $A$  is finitely  $G$ -determined if and only if the embedding  $T_A(GA) \hookrightarrow M_{m,n}$  is of finite codimension, i.e.*

$$\dim_K (M_{m,n}/T_A(GA)) < \infty.$$

*Proof.* (of the theorem) 1. We prove that

$$\mathfrak{m}^l \cdot M_{m,n} \subset \tilde{T}_A(GA).$$

Indeed, let  $B \in \mathfrak{m}^l \cdot M_{m,n}$ . Since  $A$  is  $G$   $(l-1)$ -determined,  $A + t \cdot B \in GA$  for all  $t \in K$  so that

$$\text{jet}_l(A) + t \cdot \text{jet}_l(B) \in G^{(l)} \text{jet}_l(A)$$

for all  $t \in K$ . This implies

$$\text{jet}_l(B) \in T_{\text{jet}_l(A)} \left( G^{(l)} \text{jet}_l(A) \right) = \tilde{T}_{\text{jet}_l(A)} \left( G^{(l)} \text{jet}_l(A) \right),$$

where the equality follows by the separability assumption using Corollary 2.3. Let  $p_l : M_{m,n} \rightarrow M_{m,n}^{(l)}$  be the projection. Then by taking the preimages, we obtain

$$B \in \tilde{T}_A(GA) + \mathfrak{m}^{l+1} \cdot M_{m,n}.$$

This shows that

$$\mathfrak{m}^l \cdot M_{m,n} \subset \tilde{T}_A(GA) + \mathfrak{m}^{l+1} \cdot M_{m,n},$$

and implies that

$$\mathfrak{m}^l \cdot M_{m,n} \subset \tilde{T}_A(GA) \cap \mathfrak{m}^l \cdot M_{m,n} + \mathfrak{m}^{l+1} \cdot M_{m,n}.$$

Applying Nakayama's lemma to the  $R$ -submodule  $\tilde{T}_A(GA) \cap \mathfrak{m}^l \cdot M_{m,n}$  of the  $R$ -module  $\mathfrak{m}^l \cdot M_{m,n}$  we obtain

$$\mathfrak{m}^l \cdot M_{m,n} \subset \tilde{T}_A(GA).$$

2. Follows from 1. and Proposition 4.2. □

We finish with a result for  $G = \mathcal{R} = \text{Aut}(K[[x_1, \dots, x_s]])$ , i.e. about right equivalence, by showing that finite determinacy is rather rare for  $m > 1$  under the separability condition. Let  $A \in \mathfrak{m} \cdot M_{m,n}$ . Since the action of  $G$  does not use the matrix structure, we may assume that  $A = [a_1 \ a_2 \ \dots \ a_m]^T \in M_{m,1}$  is the matrix of one column, i.e. without loss of generality we may assume that  $n = 1$ , just to make the notation shorter. Let  $\text{Jac}(A) := \left[ \frac{\partial a_i}{\partial x_j} \right] \in \text{Mat}(m, s, R)$  be the Jacobian matrix of the vector  $(a_1, \dots, a_m) \in R^m$ . Then  $\text{Jac}(A)$  is a presentation matrix of  $M_{m,1}/\tilde{T}_A^e(GA) = R^m / \langle \frac{\partial A}{\partial x_j} \rangle$  and we have:

**Proposition 4.5.** Let  $G = \mathcal{R}$  and  $A \in \mathfrak{m} \cdot M_{m,1}$ .

1. If  $m > s$  then

$$\dim_K \left( M_{m,1}/\tilde{T}_A(GA) \right) = \infty.$$

If in addition there is some  $k \in \mathbb{N}$  such that the orbit map  $G^{(l)} \rightarrow G^{(l)} \text{jet}_l(A)$  is separable for all  $l \geq k$ , then  $A$  is not finitely right determined.

2. If  $m \leq s$  and  $\{\overline{a_i}\}$  is linearly independent in  $\mathfrak{m}/\mathfrak{m}^2$ , then  $A$  is finitely right determined. If  $m > 1$  and there is some  $k \in \mathbb{N}$  such that the orbit map  $G^{(l)} \rightarrow G^{(l)} \text{jet}_l(A)$  is separable for all  $l \geq k$ , then the converse also holds.

*Proof.* 1. Since  $I_m(\text{Jac}(A)) = \{0\}$ , by Lemma 4.1  $\tilde{T}_A(GA)$  has infinite codimension. If the assumption on separability is satisfied then by Theorem 4.3,  $A$  is not finitely right determined.

2. Indeed, we prove that  $A$  is right 1-determined in the first assertion. Let  $B = [b_1 \ b_2 \ \dots \ b_m]^T \in M_{m,1}$  be such that  $c_i := b_i - a_i \in \mathfrak{m}^2$  for all  $i = 1, \dots, m$ . Let  $a_{m+1}, \dots, a_s \in \mathfrak{m}$  be such that  $\{\overline{a_i}\}$ ,  $i = 1, \dots, s$  is a basis of  $\mathfrak{m}/\mathfrak{m}^2$ . Let  $\phi_1, \phi_2 : R \rightarrow R$  be automorphisms defined by  $\phi_1(x_\nu) = a_\nu$  for  $\nu = 1, \dots, s$ , and  $\phi_2(x_\nu) = a_\nu + c_\nu$  for  $\nu = 1, \dots, m$  and  $\phi_2(x_\nu) = a_\nu$  for  $\nu = m+1, \dots, s$ . Then  $\phi := \phi_2 \circ \phi_1^{-1} \in \mathcal{R}$  and  $\phi(A) = B$ . For the second statement, assume that  $A$  is finitely right determined. By the assumption on separability of the orbit maps and the statement 1. we have  $m \leq s$ . Moreover, assume by contradiction that  $\{\overline{a_i}\}$  is linearly dependent in  $\mathfrak{m}/\mathfrak{m}^2$ . Then  $\text{rank} \left[ \frac{\partial a_i}{\partial x_j}(\mathbf{0}) \right] < m$ , where  $\frac{\partial a_i}{\partial x_j}(\mathbf{0})$  is the constant term of  $\frac{\partial a_i}{\partial x_j}$ . This implies that  $I_m(\text{Jac}(A))$  is a proper ideal of  $R$ , and by [Mat89, Theorem 13.10], this ideal has the height at most  $s - m + 1$ . This yields

$$\dim(R/I_m(\text{Jac}(A))) \geq m - 1 > 0$$

so that  $I_m(\text{Jac}(A))$  can not contain any power of  $\mathfrak{m}$ . By Proposition 4.3,  $A$  is not finitely right determined, a contradiction.  $\square$

**Corollary 4.6.** If  $\text{char}(K) = 0$  we have

- (i) If  $m > s$  then  $A$  is not finitely right determined.

- (ii) Let  $m \leq s$ . If  $m > 1$  then  $A$  is finitely right determined if and only if  $\{\overline{a_i}\}$  is linearly independent in  $\mathfrak{m}/\mathfrak{m}^2$ .

**Remark 4.7.** If  $m = 1$  and the characteristic of  $K$  is arbitrary, then  $A = [a]$  is finitely right determined if and only if  $M_{m,n}/\tilde{T}_A^e(GA) = R/\left\langle \frac{\partial a}{\partial x_1}, \dots, \frac{\partial a}{\partial x_s} \right\rangle$  is finite dimensional over  $K$ . This is well known if  $\text{char}(K) = 0$ . The case of positive characteristic was stated in [BGM12] but the proof contains a gap, which is closed in [GP16].

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